

Metrics On Unitary Matrices And Their Application To Quantifying The Degree Of Non-Commutativity Between Unitary Matrices

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By studying the minimum resources required to perform a unitary transformation, families of metrics and pseudo-metrics on unitary matrices that are closely related to a recently reported quantum speed limit by the author are found. Interestingly, this family of metrics can be naturally converted into useful indicators of the degree of non-commutativity between two unitary matrices.

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I. INTRODUCTION

Quantum information processing is the study of methods and efficiency in storage, manipulation and conversion of information represented by quantum states. Many quantum information theoretic concepts are closely related to geometry. For instance, trace distance and fidelity, which come out of the study of distinguishability between quantum states, are closely linked with Bures and Fubini-Study metrics. (See, for example, Ref. [1] for an overview.) A few quantum codes can be constructed by algebraic-geometric means [2]. And finding the optimal quantum circuit can be regarded as the problem of finding the shortest path between two points in a certain curved geometry [3].

Recently, a few metrics on unitary operators with quantum information applications were found. For example, Johnston and Kribs introduced the k th operator norm of an operator acting on a bipartite system by considering the action of the operator on bipartite states with Schmidt rank less than or equal to k . The k th operator norm can be used to study bound entanglement of Werner states as well as to construct several new entanglements witnesses [4, 5]. Rastegin studied the partitioned trace distance which shares similar properties with the standard trace distance [6]. The partitioned trace distance shines new light on exponential indistinguishability and hence can be used to investigate certain quantum cryptographic problems [7]. Interestingly, both the k th operator norm and the partitioned trace distance are related to the Ky Fan norm [4, 6].

By asking the question about the minimum resources needed to perform a unitary transformation (a question of quantum information processing favor), I report families of related metrics and pseudo-metrics on the set of unitary matrices (a result of geometric nature) in this paper. In this regard, these metrics and pseudo-metrics are very different from trace distance, partitioned trace

distance and k th operator norm as the latter are more closely related to a different quantum information processing problem, namely, the distinguishability of states and operators.

An interesting consequence of the discovery of this new family of metrics on the set of unitary matrices is that it gives refined measures of the degree of non-commutativity between two unitary matrices beyond the standard yes-or-no answer. Remarkably, while quantifying the level of closeness between two quantum states can be done by tools such as trace distance and fidelity (see, for example, Refs. [1, 8] for an introduction on this matter), little has been done on the quantification of the degree of non-commutativity between two unitary operators.

Actually, the metrics and pseudo-metrics reported in this paper are constructed from certain linear combinations of the absolute values of the arguments of eigenvalue of a unitary matrix. To prove that these constructions are indeed metrics and pseudo-metrics, specific inequalities concerning the absolute values of the arguments of eigenvalue of the unitary matrices U, V and UV have to be established. The precise statements to be proven can be found in Definitions 2 and 4 as well as Theorem 2 below. Note that various authors had shown the validity of a similar inequality, with the weighted sum of the absolute values of the arguments of eigenvalue for each unitary matrix being replaced by the largest argument of the eigenvalue for that matrix, provided that an additional condition constraining the arguments of eigenvalues of U and V is satisfied [9–12]. However, it is not clear how to modify their proofs to show the validity of the inequalities stated in Theorem 2. In fact, only a few results on the relationship between arguments of eigenvalues of unitary matrices U, V and UV are known. Most mathematical works along this general direction concentrate on the study of spectral variations of normal matrices on the complex plane (rather than arguments of the eigenvalues) as well as relations between the eigenvalues of Hermitian matrices H_1, H_2 and $H_1 + H_2$ (known as the Weyl's problem). (See, for example, Refs. [13–15] for comprehensive surveys.) Besides, techniques used to

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tackle the Weyl's problem, such as the min-max principle stated in Sec. III.1 of Ref. [13], cannot be easily adapted to the case of unitary matrices. In this regard, the proof of Theorem 2 is also of mathematical interest.

I begin by asking what is the minimum resources needed to perform a unitary transformation in Sec. II. Motivated by the result of this quantum information theoretic question, I define two closely related families of binary operations on the set of unitary matrices $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$, where $\vec{\mu}$ is a real-valued vector satisfying a technical condition, in Sec. III. Then, in Sec. IV, I prove that these two families of binary operations are families of metrics and pseudo-metrics on the set of all unitary matrices, respectively. In Sec. V, I apply the metric introduced in Sec. III to measure the degree of non-commutativity between two unitary matrices. Finally, I give a summary and outlook in Sec. VI.

II. MINIMUM RESOURCES NEEDED TO PERFORM A UNITARY TRANSFORMATION

Two commonly studied problems in the field of quantum information processing are the maximum efficiency of a particular quantum information processing operation as well as the minimum resources needed to carry out such operation allowed by the known laws of nature [16]. Since a unitary operator U , whose eigenvalues can be written in the form $e^{i\theta_j}$'s with $\theta_j \in (-\pi, \pi]$, is the result of time evolution of an Hamiltonian H , one may naively use the values of $|\theta_j|$'s as indicators of the minimum resources (in terms of the product of the average energy of the system and the evolution time needed) required to implement U . Nonetheless, this idea has to be polished as there is no physical meaning for the reference energy level and the overall phase of a unitary operator has no effect when applied to a density matrix.

To refine the above idea, I use the following result.

Definition 1. Let H be an n -dimensional Hermitian matrix. I follow the notation in Ref. [13] by denoting the eigenvalues and singular values of H arranged in descending order by $\lambda_j^{\downarrow}(H)$'s and $s_j^{\downarrow}(H)$'s respectively, where the index j runs from 1 to n . I denote the normalized eigenvector of H with eigenvalue $\lambda_j^{\downarrow}(H)$ by $|\xi_j^{\downarrow}(H)\rangle$.

Recently, I showed that given a time-independent Hamiltonian H and a normalized initial state $|\phi\rangle = \sum_j \alpha_j |\xi_j^{\downarrow}(H)\rangle$, the time τ needed to evolve $|\phi\rangle$ to a state in its orthogonal subspace satisfies [17]

$$\tau \geq \frac{\hbar}{A \sum_j |\alpha_j|^2 |\lambda_j^{\downarrow}(H) - x|} \quad (1)$$

for any $x \in \mathbb{R}$. Here A is a universal positive constant independent of H and $|\phi\rangle$. Note that $\sum_j |\alpha_j|^2 |\lambda_j^{\downarrow}(H) - x|$ is minimized when x equals the median energy of the system M . Recall that the median energy M satisfies

$$\sum_{j: \lambda_j^{\downarrow}(H) \geq M} |\alpha_j|^2 \geq \frac{1}{2} \quad (2a)$$

and

$$\sum_{j: \lambda_j^{\downarrow}(H) \leq M} |\alpha_j|^2 \geq \frac{1}{2}. \quad (2b)$$

So, M need not be unique; and any M obeying the above two equations can minimize $\sum_j |\alpha_j|^2 |\lambda_j^{\downarrow}(H) - x|$. From Eq. (1), I get

$$\tau \geq \frac{\hbar}{A \sum_j |\alpha_j|^2 |\lambda_j^{\downarrow}(H) - M|} \equiv \frac{\hbar}{A \mathcal{D}E(H, |\phi\rangle)} \quad (3)$$

where $\mathcal{D}E(H, |\phi\rangle)$ is the so-called average absolute deviation from the median of the energy of the system. Since Eq. (3) turns out to be the best possible lower bound on τ given only $\mathcal{D}E(H, |\phi\rangle)$, I interpreted $\mathcal{D}E(H, |\phi\rangle)$ as an indicator of the maximum quantum information processing rate of a system [17].

The above result can be used to prove the theorem below which quantifies the resources needed to perform a unitary transformation. The detailed proof can be found in Appendix A.

Theorem 1. Let $(\alpha_j)_{j=1}^n$ be a sequence of complex numbers obeying $\sum_{j=1}^n |\alpha_j|^2 = 1$ and $|\alpha_1|^2 \geq |\alpha_2|^2 \geq \dots \geq |\alpha_n|^2$. And let U be a given $n \times n$ unitary matrix. Then

$$R = \min_{x \in [0, 2\pi)} \min_{Ht: \exp(-iHt/\hbar) = e^{ix}U} \max_{|\phi\rangle \in C(H, (\alpha_j))} \frac{A \mathcal{D}E(H, |\phi\rangle)t}{\hbar} \quad (4)$$

exists, where $C(H, (\alpha_j))$ is the set of normalized state kets in the form $\sum_{j=1}^n \alpha_j |\xi_{P(j)}^{\downarrow}(H)\rangle$ for some permutation P of $\{1, 2, \dots, n\}$. Moreover, the H and $|\phi\rangle$ that attain the extremum in Eq. (4) can be chosen to have 0 median system energy and $\lambda_j^{\downarrow}(H)t/\hbar \in (-\pi, \pi]$ for all j .

In particular, there exists a Hamiltonian H such that

$$R = \frac{At \sum_{j=1}^n |\alpha_j|^2 s_j^{\downarrow}(H)}{\hbar}. \quad (5)$$

Theorem 1 can be understood physically as follows. Recall that $\mathcal{D}E(H, |\phi\rangle)$ is an indicator of the maximum quantum information processing rate. Since $U = e^{-iHt/\hbar}$ up to an overall phase, one may implement U , for instance, by evolving the system with a slow quantum information processing rate H for a long time or alternatively by evolving the system with a fast quantum information processing rate H for a short time. In this respect, it is the product of $\mathcal{D}E(H, |\phi\rangle)$ and evolution time t that characterizes the resources needed to implement U . More precisely, the value of R defined by Eq. (4) in Theorem 1 can be regarded as an indicator of the least amount of resources needed to carry out the unitary transformation U on the worst possible $|\phi\rangle$ in the form $\sum \alpha_j |\xi_{P(j)}^\downarrow(H)\rangle$ for some permutation P of $\{1, 2, \dots, n\}$. The larger the value of R , the more the average absolute deviation from the median of the energy of the system times the evolution time is needed to carry out the transformation on the worst possible $|\phi\rangle$.

III. DEFINING FAMILIES OF QUANTUM INFORMATION THEORY INSPIRED METRICS AND PSEUDO-METRICS ON THE SET OF UNITARY MATRICES

Based on the quantum information theoretic analysis in Sec. II and the fact that R in Eqs. (4) and (5) are essentially a weighted sum of singular values of the operator $i \log U - xI$ for some $x \in \mathbb{R}$, I propose a number of measures of the minimum amount of the average absolute deviation from the median of the energy of the system times the evolution time required to implement a unitary operation. I begin by introducing a few notations.

Definition 2. Let U be an n -dimensional unitary matrix. Generalizing the convention adopted in Ref. [13], I denote the arguments (from now on, all arguments in this paper are in principal values unless otherwise stated) of the eigenvalues of U arranged in descending order by $\theta_j^\downarrow(U)$'s (where the index j runs from 1 to n). That is to say, $U = \sum_j e^{i\theta_j^\downarrow(U)} |\phi_j^\downarrow(U)\rangle \langle \phi_j^\downarrow(U)|$ where $\theta_j^\downarrow(U) \in (-\pi, \pi]$ and $|\phi_j^\downarrow(U)\rangle$ is a normalized eigenvector of U with eigenvalue $e^{i\theta_j^\downarrow(U)}$. Similarly, I denote by $|\theta_j^\downarrow(U)\rangle$'s the absolute values of the arguments of the eigenvalue of U arranged in descending order. Occasionally, I need to refer to the arguments of the eigenvalues of U without any specific order using the notation θ_j^U 's. And in this case, I denote the corresponding normalized eigenvector by $|\phi_j^U\rangle$.

For example, let $U = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$. Then, $\theta_1^\downarrow(U) = 0$, $\theta_2^\downarrow(U) = -\pi/2$, $|\theta_1^\downarrow(U)| = \pi/2$ and $|\theta_2^\downarrow(U)| = 0$.

Note further that for any n -dimensional unitary matrix U ,

$$\theta_j^\downarrow(U^{-1}) = -\theta_{n-j+1}^\downarrow(U), \quad (6a)$$

$$|\theta_j^\downarrow(U^{-1})| = |\theta_j^\downarrow(U)|, \quad (6b)$$

and

$$\theta_j^\downarrow(U) \leq |\theta_j^\downarrow(U)|. \quad (6c)$$

Definition 3. Let $a \in \mathbb{R}$ and $U = \sum_j e^{ia\theta_j^\downarrow(U)} |\phi_j^\downarrow(U)\rangle \langle \phi_j^\downarrow(U)|$ be a unitary matrix. Then, U^a is defined to be the unitary matrix $\sum_j e^{ia\theta_j^\downarrow(U)} |\phi_j^\downarrow(U)\rangle \langle \phi_j^\downarrow(U)|$.

Definition 4. Let U, V be two n -dimensional unitary matrices. Let $\vec{\mu} = (\mu_1, \mu_2, \dots, \mu_n) \neq \vec{0}$ with $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$. I define

$$\nu(U)_{\vec{\mu}} = \sum_{j=1}^n \mu_j |\theta_j^\downarrow(U)|, \quad (7)$$

$$d_{\vec{\mu}}(U, V) = \nu(UV^{-1})_{\vec{\mu}}, \quad (8)$$

$$\nu(U)_{\vec{\mu}}^\nabla = \min_{x \in \mathbb{R}} \nu(e^{ix} U)_{\vec{\mu}}, \quad (9)$$

and

$$d_{\vec{\mu}}^\nabla(U, V) = \min_{x \in \mathbb{R}} d_{\vec{\mu}}(e^{ix} U, V). \quad (10)$$

I am going to prove in Sec. IV that $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^\nabla(\cdot, \cdot)$ are a metric and pseudo-metric, respectively. This justifies the use of the notations. On the other hand, $\nu(\cdot)_{\vec{\mu}}$ and $\nu(\cdot)_{\vec{\mu}}^\nabla$ are not a norm or pseudo-norm. This is because $U + V$ may not be unitary so that $\nu(U + V)_{\vec{\mu}}$ and $\nu(U + V)_{\vec{\mu}}^\nabla$ are ill-defined. Hence, it does not make sense to talk about validity of the (additive) triangle inequality for $\nu(\cdot)_{\vec{\mu}}$ and $\nu(\cdot)_{\vec{\mu}}^\nabla$. Nevertheless, I am going to prove in Sec. IV that $\nu(\cdot)_{\vec{\mu}}$ and $\nu(\cdot)_{\vec{\mu}}^\nabla$ satisfy the multiplicative triangle inequality.

Note further that $\nu(U)_{\vec{\mu}}$ is a weighted sum of the singular values of a certain infinitesimal generator of the unitary operator U . And from Theorem 1, I know that $\nu(U)_{\vec{\mu}}^\nabla$ is indeed a measure of the minimum average absolute deviation from the median energy of the system times the evolution time needed to perform the unitary transformation U in the sense that

$$\nu(U)_{\vec{\mu}}^\nabla = \min_{x \in [0, 2\pi)} \min_{Ht: \exp(-iHt/\hbar) = e^{ix} U} \max \mathcal{D}E(H, |\phi\rangle) t, \quad (11)$$

where the maximum is taken over all state kets $|\phi\rangle$ in the form $\sum_{j=1}^n \alpha_j |\xi_{P(j)}^\downarrow(H)\rangle$ with $|\alpha_j|^2 = \mu_j / \sum_k \mu_k$ and P is a permutation of $\{1, 2, \dots, n\}$. In contrast, $\nu(U)_{\vec{\mu}}$ can be regarded as an “unoptimized” version of $\nu(U)_{\vec{\mu}}^\nabla$ in the sense that

$$\nu(U)_{\vec{\mu}} = \min_{Ht: \exp(-iHt/\hbar) = U} \max \mathcal{D}E(H, |\phi\rangle) t, \quad (12)$$

where the maximum is taken over the same set as in the maximum in Eq. (11). In fact, the global phase of U , which has no physical meaning, affects the value of $\nu(U)_{\vec{\mu}}$ but not $\nu(U)_{\vec{\mu}}^{\nabla}$. Nevertheless, readers will find out in Sec. V that $\nu(\cdot)_{\vec{\mu}}$ is useful in studying the degree of non-commutativity between two unitary operators.

Three important points are stated. First, one should investigate the properties of Eqs. (7)–(10) for a general $\vec{\mu}$ in order to get a more complete picture on the minimum resources needed to evolve different kind of initial states $|\phi\rangle$'s by the unitary operator U .

Second, an inequality giving a lower evolution time bound like the one stated in Eq. (3) is sometimes known as a quantum speed limit [16]. In addition to Eq. (3), the other two important quantum speed limits are the so-called time-energy uncertainty relation $\tau \geq \pi\hbar/2\Delta E$ where ΔE is the standard deviation of the energy of the system and the Margolus-Levitin bound $\tau \geq \pi\hbar/2(E - E_0)$ where E and E_0 are the average energy and ground state energy of the system, respectively [18, 19]. One may define similar measures of the minimum resources needed to implement a unitary operator U based on these two quantum speed limits. Unfortunately, it is not very fruitful to pursue in this direction because these measures are not metrics. (Note that the absence of metric structures here only refers to the measure of minimum resources required to perform a unitary operation based on these quantum speed limits. It does not mean that the two quantum speed limits have no geometrical meaning. In fact, the time-energy uncertainty relation itself is closely related to the Bures metric [20].)

Third, $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ is related to the distance between two quantum states as follows. Suppose $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are two n -leveled pure states. Recall that the Bures angle between them is $\chi = |\langle\Psi_1|\Psi_2\rangle|$. Then, amongst all the unitary transformations U obeying $U|\Psi_1\rangle = |\Psi_2\rangle$, the one that minimizes $\nu(U)_{\vec{\mu}}^{\nabla}$ satisfies $\theta_1^{\dagger}(U) = \chi$, $\theta_n^{\dagger}(U) = -\chi$ and $\theta_j^{\dagger}(U) = 0$ for $1 < j < n$. (One quick way to see this is that the required U only need to perform rotation on the Hilbert space spanned by $|\Psi_1\rangle$ and $|\Psi_2\rangle$. Then the problem can be handled by standard minimizing techniques.)

IV. PROPERTIES OF $\nu(\cdot)_{\vec{\mu}}$, $d_{\vec{\mu}}(\cdot, \cdot)$, $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ AND $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$

Since $\vec{\mu} \equiv (\mu_1, \mu_2, \dots, \mu_n) = \sum_{j=1}^{n-1} (\mu_j - \mu_{j+1})\vec{\mu}^{[j]} + \mu_n\vec{\mu}^{[n]}$ where

$$\vec{\mu}^{[j]} = (\overbrace{1, 1, \dots, 1}^{j \text{ entries}}, 0, 0, \dots, 0), \quad (13)$$

I conclude that

$$\nu(U)_{\vec{\mu}} = \sum_{j=1}^{n-1} (\mu_j - \mu_{j+1})\nu(U)_{\vec{\mu}^{[j]}} + \mu_n\nu(U)_{\vec{\mu}^{[n]}} \quad (14a)$$

and

$$\nu(U)_{\vec{\mu}}^{\nabla} \geq \sum_{j=1}^{n-1} (\mu_j - \mu_{j+1})\nu(U)_{\vec{\mu}^{[j]}}^{\nabla} + \mu_n\nu(U)_{\vec{\mu}^{[n]}}^{\nabla}. \quad (14b)$$

Furthermore, the coefficients in the R.H.S. of the above two equations are non-negative. It turns out that one can deduce many properties of $\nu(\cdot)_{\vec{\mu}}$ from the properties of $\nu(\cdot)_{\vec{\mu}^{[j]}}$'s.

Here I list a few basic properties of $\nu(\cdot)_{\vec{\mu}}$, $\nu(\cdot)_{\vec{\mu}}^{\nabla}$, $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$. The proofs are left to the readers as they are straightforward. (Note that all U 's and V 's appear below are n -dimensional unitary matrices.)

- $\nu(\cdot)_{\vec{\mu}}$ and $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ are both functions from $U(n)$, the set of all $n \times n$ unitary matrices, to $[0, \pi \sum_{j=1}^n \mu_j]$. In fact, $\nu(\cdot)_{\vec{\mu}}$ is a surjection with $\nu(U)_{\vec{\mu}} = \pi \sum_j \mu_j$ for all $\vec{\mu}$ if and only if $U = -I$.
- $\nu(U)_{\vec{\mu}} = 0$ and $\nu(U)_{\vec{\mu}}^{\nabla} = 0$ if and only if $U = I$ and $U = e^{ix}I$ for some $x \in \mathbb{R}$, respectively.
- $\nu(e^{ix}U)_{\vec{\mu}}^{\nabla} = \nu(U)_{\vec{\mu}}^{\nabla}$ for all $x \in \mathbb{R}$.
- $\nu(U^{-1})_{\vec{\mu}} = \nu(U)_{\vec{\mu}} = \nu(VUV^{-1})_{\vec{\mu}}$ and $\nu(U^{-1})_{\vec{\mu}}^{\nabla} = \nu(U)_{\vec{\mu}}^{\nabla} = \nu(VUV^{-1})_{\vec{\mu}}^{\nabla}$.
- $\nu(U)_{a\vec{\mu}} = a\nu(U)_{\vec{\mu}}$ and $\nu(U)_{a\vec{\mu}}^{\nabla} = a\nu(U)_{\vec{\mu}}^{\nabla}$ for all $a > 0$.
- $\nu(U^b)_{\vec{\mu}} \leq |b|\nu(U)_{\vec{\mu}}$ and $\nu(U^b)_{\vec{\mu}}^{\nabla} \leq |b|\nu(U)_{\vec{\mu}}^{\nabla}$ for all $b \in \mathbb{R}$. Moreover, the equalities hold if $|b||\theta_1^{\dagger}(U)| \leq \pi$.
- $\nu(U)_{\vec{\mu}^{[2]}}^{\nabla} = 2\nu(U)_{\vec{\mu}^{[1]}}^{\nabla}$ and $\nu(U)_{\vec{\mu}^{[2j+1]}}^{\nabla} = \nu(U)_{\vec{\mu}^{[2j]}}^{\nabla}$ whenever $2j + 1 \leq n$.
- If $U(t)$ is a continuous one-parameter family of unitary matrices, then $\nu(U(t))_{\vec{\mu}}$ and $\nu(U(t))_{\vec{\mu}}^{\nabla}$ are continuous. This result is more involved. By the theorem that roots of a polynomial vary continuously with its coefficients [21, 22], eigenvalues of $U(t)$ must vary continuously on the unit circle. And as the absolute value of argument of a complex-valued and nowhere zero function is continuous, $\nu(U(t))_{\vec{\mu}}$ is continuous. The continuity of $\nu(U(t))_{\vec{\mu}}^{\nabla}$ then follows the fact that the pointwise minimum of a collection of continuous functions is also a continuous function if it exists.
- If $H(t)$ is an n -dimensional Hamiltonian and $U(t)$ is the unitary operator generated by $H(t)$, then

$$\left. \frac{d\nu(U(t))_{\vec{\mu}}}{dt} \right|_{t=0} = \sum_{j=1}^n \mu_j s_j^{\dagger}(H(0)). \quad (15)$$

This shows the relation between rate of change of $\nu(U(t))_{\vec{\mu}}$ and the singular values of its generator

$H(t)$. Actually, the R.H.S. of Eq. (15) is the so-called generalized spectral norm [23] of the Hamiltonian $H(0)$.

- $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$ are positive definite and positive semi-definite functions, respectively. And they are both symmetric functions.
- $d_{\vec{\mu}}(U, V) = \pi \sum_j \mu_j$ for all $\vec{\mu}$ if and only if $V = -U$.
- Let U_i be n_i -dimensional unitary matrices, then $\nu(U_1 \otimes U_2)_{\vec{\mu}^{[1]}} \leq \nu(U_1)_{\vec{\mu}^{[1]}} + \nu(U_2)_{\vec{\mu}^{[1]}}$, $\nu(U_1 \otimes U_2)_{\vec{\mu}^{[1]}}^{\nabla} \leq \nu(U_1)_{\vec{\mu}^{[1]}}^{\nabla} + \nu(U_2)_{\vec{\mu}^{[1]}}^{\nabla}$, $\nu(U_1 \otimes U_2)_{\vec{\mu}^{[n_1 n_2]}} \leq n_2 \nu(U_1)_{\vec{\mu}^{[n_1]}} + n_1 \nu(U_2)_{\vec{\mu}^{[n_2]}}$ and $\nu(U_1 \otimes U_2)_{\vec{\mu}^{[n_1 n_2]}}^{\nabla} \leq n_2 \nu(U_1)_{\vec{\mu}^{[n_1]}}^{\nabla} + n_1 \nu(U_2)_{\vec{\mu}^{[n_2]}}^{\nabla}$. (Note that I have abused the notation a little bit as clearly the lengths of the vectors $\vec{\mu}^{[1]}$, $\vec{\mu}^{[n_1]}$, $\vec{\mu}^{[n_2]}$, $\vec{\mu}^{[n_1 n_2]}$ in the above inequalities are different even though each of them has the same number of non-zero entries.)

Interestingly, except for the second property (the necessary and sufficient condition for $\nu(\cdot)_{\vec{\mu}}$ and $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ to be zero) and the last property (concerning the tensor product of unitary matrices), all the above properties require only $\mu_j \geq 0$ for all j . In contrast, the condition $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$ is required for $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$ to satisfy the triangle inequality. In fact, the triangle inequality follows directly from Eq. (8) and the theorem below. And this theorem alone is also of great interest as it bounds certain weighted sums of the arguments of eigenvalues for the product of two unitary matrices.

Theorem 2. *Let U, V be two n -dimensional unitary matrices and $\vec{\mu}$ satisfying the requirements stated in Definition 4. Then*

$$|\nu(U)_{\vec{\mu}} - \nu(V)_{\vec{\mu}}| \leq \nu(UV)_{\vec{\mu}} \leq \nu(U)_{\vec{\mu}} + \nu(V)_{\vec{\mu}}. \quad (16)$$

I move both the proof of Theorem 2 and the conditions for the second inequality in Eq. (16) to become an equality to Appendix B as they are rather involved and technical.

Corollary 1. *Eq. (16) in Theorem 2 also holds if $\nu(\cdot)_{\vec{\mu}}$ is replaced by $\nu(\cdot)_{\vec{\mu}}^{\nabla}$.*

Proof. For any n -dimensional unitary matrices U and V , there exist $x, y \in \mathbb{R}$ such that $\nu(e^{ix}U)_{\vec{\mu}} = \nu(U)_{\vec{\mu}}^{\nabla}$ and $\nu(e^{iy}V)_{\vec{\mu}} = \nu(V)_{\vec{\mu}}^{\nabla}$. By Theorem 2, $\nu(U)_{\vec{\mu}}^{\nabla} + \nu(V)_{\vec{\mu}}^{\nabla} \geq \nu(e^{i(x+y)}UV)_{\vec{\mu}} \geq \nu(UV)_{\vec{\mu}}^{\nabla}$. The proof of $|\nu(U)_{\vec{\mu}}^{\nabla} - \nu(V)_{\vec{\mu}}^{\nabla}| \leq \nu(UV)_{\vec{\mu}}^{\nabla}$ is then a replica of that of the first half of Eq. (16). \square

From Theorem 2 and Corollary 1, I conclude that $d_{\vec{\mu}}(\cdot, \cdot)$ and $d_{\vec{\mu}}^{\nabla}(\cdot, \cdot)$ are a metric and pseudo-metric, respectively. Furthermore, $\nu(\cdot)_{\vec{\mu}}$, $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ behave somewhat

like a norm and semi-norm respectively in the sense that $\nu(UV)_{\vec{\mu}} \leq \nu(U)_{\vec{\mu}} + \nu(V)_{\vec{\mu}}$. In other words, the addition operation in the conventional definition of a norm is replaced by multiplication. Note further that $\nu(\cdot)_{\vec{\mu}}$ does not obey $\nu(aU)_{\vec{\mu}} = |a|\nu(U)_{\vec{\mu}}$ for all scalars a .

Remark 1. *For any n -dimensional matrices U and V satisfying $\theta_1^{\dagger}(U) + \theta_1^{\dagger}(V) \leq \pi$ and $\theta_n^{\dagger}(U) + \theta_n^{\dagger}(V) > -\pi$, the following closely related bound*

$$\theta_1^{\dagger}(UV) \leq \theta_1^{\dagger}(U) + \theta_1^{\dagger}(V) \quad (17)$$

has been proven by a few authors using different methods [9, 11, 12]. Recently, our group [24, 25] also found elementary proofs of Eq. (17) as well as the special case of Theorem 2 in which $\vec{\mu} = \vec{\mu}^{[1]}$.

Remark 2. *The proof for the special case of $\vec{\mu} = \vec{\mu}^{[n]}$ is rather simple. I simply need to use the identity $\prod_{j=1}^n e^{i\theta_j^{\dagger}(UV)/2n} = \det[(UV)^{1/2n}] = [\det(UV)]^{1/2n} = [\det U \det V]^{1/2n} = \det U^{1/2n} \det V^{1/2n} = \prod_{j=1}^n e^{i[\theta_j^{\dagger}(U) + \theta_j^{\dagger}(V)]/2n}$. Then, by equating the arguments on both sides and by observing that both arguments are in the interval $(-\pi, \pi]$, I obtain the required inequality.*

Remark 3. *It is easy to check from the proof of Corollary 1 that $\nu(UV)_{\vec{\mu}}^{\nabla} = \nu(U)_{\vec{\mu}}^{\nabla} + \nu(V)_{\vec{\mu}}^{\nabla}$ if and only if there exist $x, y \in \mathbb{R}$ such that $\nu(U)_{\vec{\mu}}^{\nabla} = \nu(e^{ix}U)_{\vec{\mu}}$, $\nu(V)_{\vec{\mu}}^{\nabla} = \nu(e^{iy}V)_{\vec{\mu}}$, $\nu(e^{i(x+y)}UV)_{\vec{\mu}} = \nu(e^{ix}U)_{\vec{\mu}} + \nu(e^{iy}V)_{\vec{\mu}}$ and $\nu(UV)_{\vec{\mu}}^{\nabla} = \nu(e^{i(x+y)}UV)_{\vec{\mu}}$. Therefore, the conditions for equality of the triangle inequality for $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ seems to be more stringent than those for $\nu(\cdot)_{\vec{\mu}}$. On the other hand, as I have pointed out in the list of the basic properties of $\nu(\cdot)_{\vec{\mu}}^{\nabla}$ that $\nu(U)_{\vec{\mu}^{[3]}}^{\nabla} = \nu(U)_{\vec{\mu}^{[2]}}^{\nabla} = 2\nu(U)_{\vec{\mu}^{[1]}}^{\nabla}$. Thus, the conditions for $\nu(UV)_{\vec{\mu}^{[m]}}^{\nabla} = \nu(U)_{\vec{\mu}^{[m]}}^{\nabla} + \nu(V)_{\vec{\mu}^{[m]}}^{\nabla}$ is not particularly more stringent than those for $\nu(\cdot)_{\vec{\mu}^{[m]}}$ for $m \leq 3$. But in any case, as m or n increases, it is more and more difficult for $\nu(UV)_{\vec{\mu}}^{\nabla} = \nu(U)_{\vec{\mu}}^{\nabla} + \nu(V)_{\vec{\mu}}^{\nabla}$ provided that U and V are drawn randomly from the Haar measure of $U(n)$. This finding is reflected in the plots in Fig. 2. Interestingly, by comparing Figs. 1 and 2, I find that on average (over the Haar measure of $U(n)$) the value of $\nu(U)_{\vec{\mu}}^{\nabla} + \nu(V)_{\vec{\mu}}^{\nabla} - \nu(UV)_{\vec{\mu}}^{\nabla}$ is smaller than $\nu(U)_{\vec{\mu}} + \nu(V)_{\vec{\mu}} - \nu(UV)_{\vec{\mu}}$. This is probably due to the fact that $\nu(U)_{\vec{\mu}}^{\nabla} \leq \nu(U)_{\vec{\mu}}$.*

Remarkably, several new inequalities involving the eigenvalues of a product of two unitary matrices can be found by modifying the proof of Theorem 2 [26]. But I will not further elaborate on this matter here as it is beyond the scope of this paper.

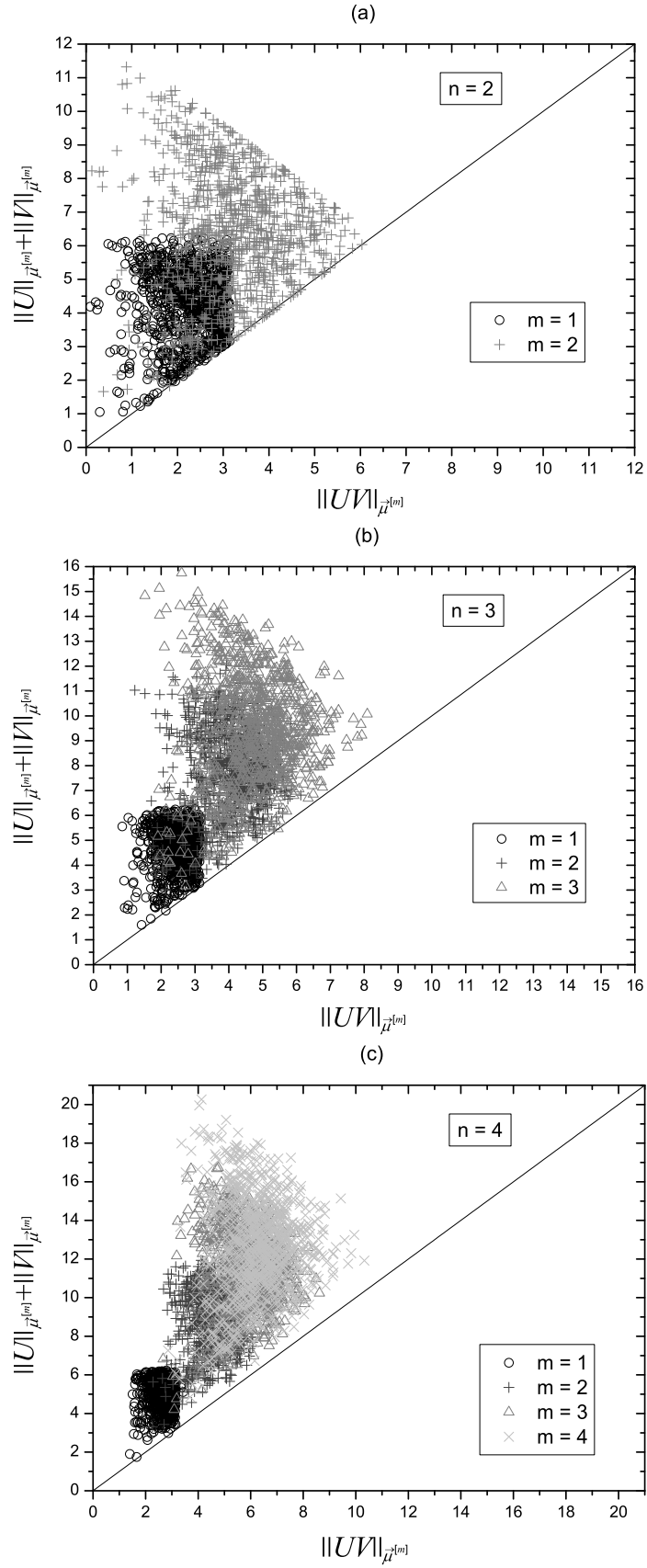


FIG. 1. Plots of $\nu(UV)_{\vec{\mu}^{[m]}}$ against $\nu(U)_{\vec{\mu}^{[m]}} + \nu(V)_{\vec{\mu}^{[m]}}$ for various values of m where the dimension n of the unitary matrices U and V equals (a) $n = 2$, (b) $n = 3$ and (c) $n = 4$. The matrices U and V are randomly chosen from the Haar measure of $U(n)$ and the sample size in each case is 1000. The black lines represent the situation in which $\nu(UV)_{\vec{\mu}^{[m]}} = \nu(U)_{\vec{\mu}^{[m]}} + \nu(V)_{\vec{\mu}^{[m]}}$.

V. MEASURE OF THE DEGREE OF NON-COMMUTATIVITY BETWEEN TWO UNITARY OPERATORS

Although the family of $\nu(\cdot)_{\vec{\mu}}$'s are only “unoptimized” measures of the minimum resources needed to perform the unitary transformation in its argument, it is useful in measuring the degree of non-commutativity between two unitary operators.

Definition 5. Let U, V two n -dimensional unitary matrices. Suppose $\vec{\mu}$ be the vector defined in Def. 4. I define

$$\mathfrak{C}_{\vec{\mu}}(U, V) = d_{\vec{\mu}}(UV, VU) \equiv \nu(UVU^{-1}V^{-1})_{\vec{\mu}}. \quad (18)$$

Physically, $\mathfrak{C}_{\vec{\mu}}(U, V)$ can be interpreted as a measure of the minimum average absolute deviation from the median energy of the system times the evolution time required to convert UV to VU . The higher the required resources, the more “non-commutative” the operators U and V are.

The following simple properties of $\mathfrak{C}_{\vec{\mu}}(\cdot, \cdot)$ carry over from the properties of $\nu(\cdot)_{\vec{\mu}}$ reported in Sec. IV:

- $\mathfrak{C}_{\vec{\mu}}(U, V) : U(n) \times U(n) \longrightarrow [0, \pi \sum_{j=1}^n \mu_j]$ is a surjective map.
- $\mathfrak{C}_{\vec{\mu}}(U, V) = 0$ if and only if U commutes with V .
- $\mathfrak{C}_{\vec{\mu}}(\cdot, \cdot)$ is not a metric. In fact, any meaningful measure of commutativity is not a metric for commutativity is not transitive.
- $\mathfrak{C}_{\vec{\mu}}(e^{ix}U, e^{iy}V) = \mathfrak{C}_{\vec{\mu}}(U, V)$ for any $x, y \in \mathbb{R}$.
- $\mathfrak{C}_{\vec{\mu}}(U^{-1}, V^{-1}) = \mathfrak{C}_{\vec{\mu}}(U, V) = \mathfrak{C}_{\vec{\mu}}(WUW^{-1}, WVW^{-1})$.
- $\mathfrak{C}_{a\vec{\mu}}(U, V) = a\mathfrak{C}_{\vec{\mu}}(U, V)$ for all $a > 0$.
- $\mathfrak{C}_{\vec{\mu}}(U, V) = \pi \sum_j \mu_j$ if $UVU^{-1}V^{-1} = -I$. In particular, the three Pauli matrices σ_x, σ_y and σ_z obey $\mathfrak{C}_{\vec{\mu}}(\sigma_x, \sigma_y) = \mathfrak{C}_{\vec{\mu}}(\sigma_y, \sigma_z) = \mathfrak{C}_{\vec{\mu}}(\sigma_z, \sigma_x) = \pi \sum_k \mu_k$. In this sense, pairs of distinct Pauli matrices are examples of the most non-commutative unitary operators in $U(2)$.
- $\mathfrak{C}_{\vec{\mu}^{[1]}}(U_1 \otimes V_1, U_2 \otimes V_2) \leq \mathfrak{C}_{\vec{\mu}^{[1]}}(U_1, V_1) + \mathfrak{C}_{\vec{\mu}^{[1]}}(U_2, V_2)$, and $\mathfrak{C}_{\vec{\mu}^{[n_1 n_2]}}(U_1 \otimes V_1, U_2 \otimes V_2) \leq n_2 \mathfrak{C}_{\vec{\mu}^{[n_1]}}(U_1, V_1) + n_1 \mathfrak{C}_{\vec{\mu}^{[n_2]}}(U_2, V_2)$.
- If $U(t)$ and $V(t)$ are continuous one-parameter families of unitary matrices, then $\mathfrak{C}_{\vec{\mu}}(U(t), V(t))$ is continuous.
- Let $H_i(t)$ be an n -dimensional Hamiltonian and $U_i(t)$ be the unitary operator generated by $H_i(t)$

for $i = 1, 2$. Then,

$$\begin{aligned} & \left. \frac{d^2 \mathfrak{C}_{\vec{\mu}}(U_1(t), U_2(t))}{dt^2} \right|_{t=0} \\ &= 2 \lim_{t \rightarrow 0} \frac{\mathfrak{C}_{\vec{\mu}}(U_1(t), U_2(t))}{t^2} \\ &= 2 \sum_{j=1}^n \mu_j s_j^\downarrow (-i[H_1(0), H_2(0)]). \end{aligned} \quad (19)$$

This shows the relation between the curvature of $\mathfrak{C}_{\vec{\mu}}(U_1(t), U_2(t))$ and the singular values of the commutator of the the corresponding generators $[H_1(t), H_2(t)]$.

VI. DISCUSSIONS

In summary, I have introduced a family of metrics $d_{\vec{\mu}}(\cdot, \cdot)$ and a family of pseudo-metrics $d_{\vec{\mu}}^\nabla(\cdot, \cdot)$ on finite-dimensional unitary matrices. In particular, the pseudo-metric $d_{\vec{\mu}}^\nabla(U, V)$ can be interpreted as a measure of the minimum average absolute deviation from the median energy of the system times the evolution time needed to perform the unitary transformation UV^{-1} (and hence equivalently also VU^{-1}). Besides, $d_{\vec{\mu}}^\nabla(\cdot, \cdot)$ is related to the Bures angle between two quantum states; while $d_{\vec{\mu}}(U, V)$ is related to the generalized spectral norm of the infinitesimal generator of UV^{-1} .

Another aspect of this paper is the proposed measure on the degree of non-commutativity between two unitary matrices U and V based on $\nu(UVU^{-1}V^{-1})_{\vec{\mu}}$. Its physical meaning is a measure of the minimum resources needed to convert UV to VU . Interestingly, this measure is related to the generalized spectral norm of the commutator of the infinitesimal generators of U and V .

The analysis here so far are restricted to finite-dimensional unitary matrices. It is instructive to see how it can be extended to cover the infinite-dimensional case. Also a possible future research direction is to investigate the possibility of extending the results here to trace-preserving completely positive maps and to find non-trivial applications in quantum information processing.

Appendix A: Proof of Theorem 1

The following lemma is needed to prove Theorem 1.

Lemma 1. Let x_j 's and y_j 's be real numbers satisfying $x_1 \geq x_2 \geq \dots \geq x_n$ and $y_1 \geq y_2 \geq \dots \geq y_n$. Then

$$\sum_{j=1}^n x_j y_{P(j)} \leq \sum_{j=1}^n x_j y_j \quad (A1)$$

for all permutation P of the set $\{1, 2, \dots, n\}$.

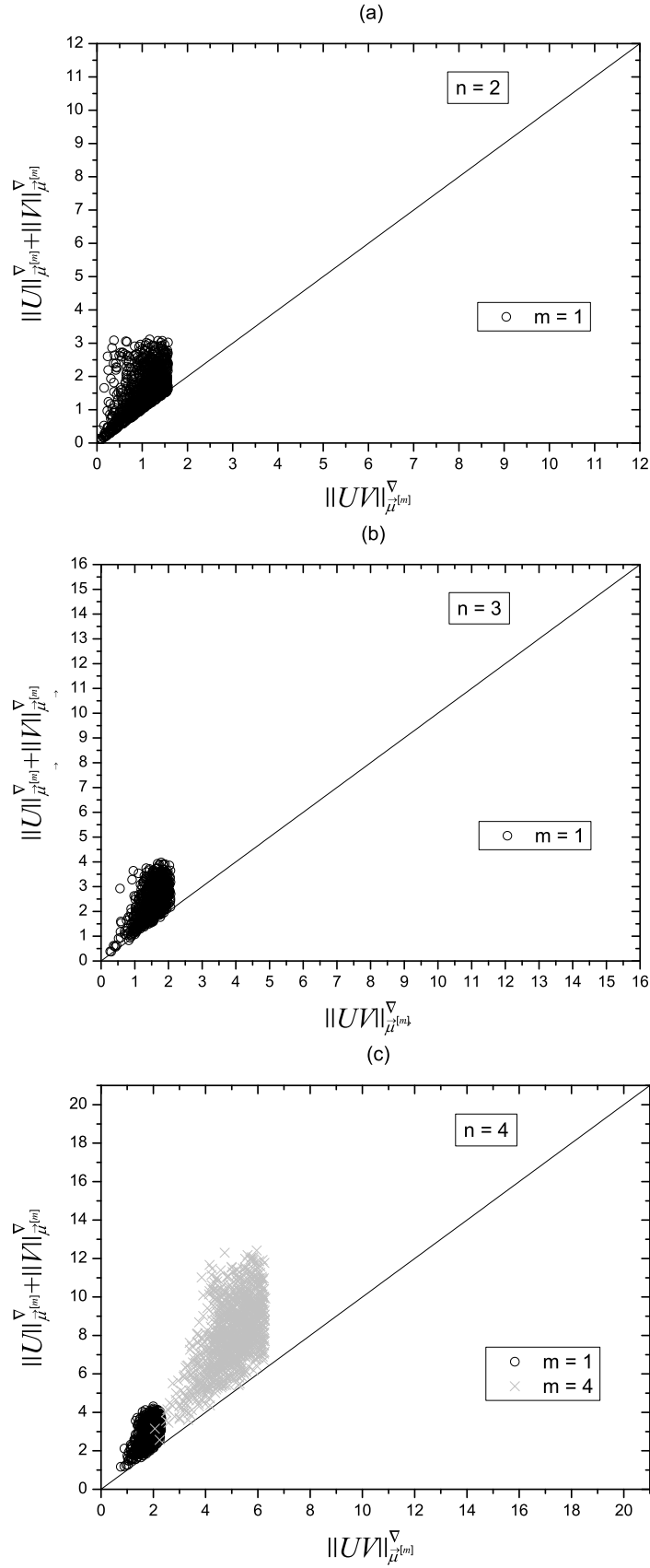


FIG. 2. Plots of $\nu(UV)_{\vec{\mu}^{[m]}}^{\nabla}$ against $\nu(U)_{\vec{\mu}^{[m]}}^{\nabla} + \nu(V)_{\vec{\mu}^{[m]}}^{\nabla}$ for various values of m where the dimension n of the unitary matrices U and V equals (a) $n = 2$, (b) $n = 3$ and (c) $n = 4$. The parameters and sampling method used are the same as those in Fig. 1. Since $\nu(U)_{\vec{\mu}^{[3]}}^{\nabla} = \nu(U)_{\vec{\mu}^{[2]}}^{\nabla} = 2\nu(U)_{\vec{\mu}^{[1]}}^{\nabla}$, I only show the cases of $m = 1$ and $m = 4$ in this figure. And to allow easy comparison, the scales used in the plots are the same as those used in Fig. 1.

Proof. Eq. (A1) is trivially true for $n = 1$. And its validity for $n = 2$ follows from the inequality $(x_1 - x_2)(y_1 - y_2) \geq 0$. This Lemma can then be proven by mathematical induction on n . \square

Proof of Theorem 1. Note that t can only be 0 when $e^{ix}U = I$. In this case, Eq. (4) clearly makes sense and is equal to 0. So, this theorem is trivially true.

$$R' = \min_{x \in [0, 2\pi)} \min_{Ht: \exp(-iHt/\hbar) = e^{ix}U} \max_{|\phi\rangle \in C(H, (\alpha_j))} \frac{A\langle\phi|\sqrt{H^\dagger H}|\phi\rangle t}{\hbar}. \quad (\text{A2})$$

Note that the set $\{|\phi|\sqrt{H^\dagger H}|\phi\rangle: |\phi\rangle \in C(H, (\alpha_j))\}$ equals $\{\sum_{j=1}^n |\alpha_j|^2 s_{P(j)}^\downarrow(H): P \in S_n\}$, where S_n denotes the permutation group of n elements. So from Lemma 1, for any given H , t and x , the maximum in Eq. (A2) exists and is equal to $At \sum_{j=1}^n |\alpha_j|^2 s_j^\downarrow(H)/\hbar$. Now suppose $s_1^\downarrow(H) \geq \pi\hbar/t$. I consider the new Hamiltonian H' formed by changing only the eigenvalue λ corresponding to $s_1^\downarrow(H)$ to $\lambda \bmod (2\pi\hbar/t) \equiv \lambda' \in (-\pi\hbar/t, \pi\hbar/t]$. Clearly, $|\lambda'| \leq |\lambda| = s_1^\downarrow(H)$ and

Whereas if $t < 0$, then $\mathcal{D}E(H, |\phi\rangle)t \leq 0$. In this case, I may consider $-H$ and $-t$ instead in analyzing Eq. (4) for $\mathcal{D}E(-H, |\phi\rangle)(-t) \geq 0$ and $\exp(-iHt/\hbar) = \exp[-i(-H)(-t)/\hbar]$. Hence, I only need to consider the remaining case of $t > 0$ from now on.

I first justify the use of maximum and minimum in Eq. (4); and I do so by considering a similar min-max expression:

$\exp(-iH't/\hbar) = \exp(-iHt/\hbar)$. In other words, there exists $k \in \{1, 2, \dots, n\}$ such that

$$s_j^\downarrow(H') = \begin{cases} s_{j+1}^\downarrow(H) & \text{if } j < k, \\ |\lambda'| & \text{if } j = k, \\ s_j^\downarrow(H) & \text{if } j > k. \end{cases} \quad (\text{A3})$$

Consequently, by Lemma 1,

$$\begin{aligned} \max_{|\phi'\rangle \in C(H', (\alpha_j))} \langle\phi'|\sqrt{H'^\dagger H'}|\phi'\rangle - \max_{|\phi\rangle \in C(H, (\alpha_j))} \langle\phi|\sqrt{H^\dagger H}|\phi\rangle &= |\alpha_k|^2 |\lambda'| - |\alpha_1|^2 s_1^\downarrow(H) + \sum_{\ell=1}^{k-1} (|\alpha_\ell|^2 - |\alpha_{\ell+1}|^2) s_{\ell+1}^\downarrow(H) \\ &\leq s_1^\downarrow(H) \left[|\alpha_k|^2 - |\alpha_1|^2 + \sum_{\ell=1}^{k-1} (|\alpha_\ell|^2 - |\alpha_{\ell+1}|^2) \right] \\ &= 0. \end{aligned} \quad (\text{A4})$$

That is to say, $\max_{|\phi\rangle \in C(H, (\alpha_j))} \langle\phi|\sqrt{H^\dagger H}|\phi\rangle$ does not increase if H is replaced by H' . By repeating this procedure at most n times, I conclude that in order to look for the minimum in Eq. (A2), I only need to consider those H 's with all their eigenvalues in $(-\pi\hbar/t, \pi\hbar/t]$. Note that the expression $\max_{|\phi\rangle \in C(H, (\alpha_j))} \langle\phi|\sqrt{H^\dagger H}|\phi\rangle$ depends on the eigenvalues of H rather than the eigenvectors of H . Moreover, for a fixed $x \in \mathbb{R}$, there is a unique set of eigenvalues $\lambda_j^\downarrow(H)$'s for H such that $\lambda_j^\downarrow(H) \in (-\pi\hbar/t, \pi\hbar/t]$ and $\exp(-iHt/\hbar) = e^{ix}U$. So, the second minimum expression (that is, the minimum over Ht) in Eq. (A2) exists for each $x \in \mathbb{R}$. As x varies, the set of eigenvalues of H that minimizes the second minimum expression in Eq. (A2) changes linearly with x modulo $2\pi\hbar/t$. Hence, this second minimum expression is a continuous function of $x \in \mathbb{R}$ with period 2π . Therefore, R' exists.

Now I go on to show that the H and $|\phi\rangle$ that minimize Eq. (A2) can be chosen to have median system energy $M = 0$. First, I claim that $\sum^- |\alpha_j|^2 \geq 1/2$ where

the sum is over all j 's with $\lambda_j^\downarrow(H)t/\hbar \in \{\pi\} \cup (-\pi, 0]$. Suppose the contrary, as H has at most n distinct eigenvalues, there exists $\epsilon < 0$ sufficiently close to 0 such that $\langle\phi|\sqrt{(H+\epsilon)^\dagger(H+\epsilon)}|\phi\rangle < \langle\phi|\sqrt{H^\dagger H}|\phi\rangle$. This contradicts the assumption that H and $|\phi\rangle$ extremize Eq. (A2). By a similar argument, $\sum^+ |\alpha_j|^2 \leq 1/2$ where the sum is over all j 's with $\lambda_j^\downarrow(H)t/\hbar \in [0, \pi)$. Hence, from Eqs. (2a) and (2b), the median energy of the system M is equal to 0.

From the discussions between Eqs. (1) and (3) on the minimization of $\sum_j |\alpha_j|^2 |\lambda_j^\downarrow(H') - x|$, I conclude that $\mathcal{D}E(H', |\phi\rangle) \leq \max_{|\phi\rangle \in C(H', (\alpha_j))} \langle\phi|\sqrt{H'^\dagger H'}|\phi\rangle$ with equality holds if the median system energy $M = 0$. By comparing Eq. (4) with Eq. (A2), I deduce that $R \leq R'$ if R exists. Recall that Eq. (A2) is well-defined and the extremum can be attained by H and $|\phi\rangle$ such that $\lambda_j^\downarrow(H) \in (-\pi\hbar/t, \pi\hbar/t]$ and the median system energy $M = 0$. With these H and $|\phi\rangle$, $\mathcal{D}E(H, |\phi\rangle) =$

$\sum_{j=1}^n |\alpha_j|^2 s_j^\dagger(H) = \langle \phi | \sqrt{H^\dagger H} | \phi \rangle$. Hence, R exists and is equal to R' . That is to say, Eq. (4) is well-defined and its extremum is attained by picking H and $|\phi\rangle$ so that $M = 0$ and $\lambda_j^\dagger(H)t/\hbar \in (-\pi, \pi]$. \square

Appendix B: Proof of Theorem 2 and the conditions for equality in Eq. (16)

Proof of Theorem 2. I only need to show the second half of the inequality in Eq. (16) as the first half follows directly from it. More precisely, from the second half of Eq. (16), $\nu(U)_{\bar{\mu}} \leq \nu(UV)_{\bar{\mu}} + \nu(V^{-1})_{\bar{\mu}} = \nu(UV)_{\bar{\mu}} + \nu(V)_{\bar{\mu}}$; and similarly $\nu(V)_{\bar{\mu}} \leq \nu(U^{-1})_{\bar{\mu}} + \nu(UV)_{\bar{\mu}} = \nu(U)_{\bar{\mu}} + \nu(UV)_{\bar{\mu}}$. And from Eq. (14a), it suffices to prove this theorem by showing its validity for all $\vec{\mu}^{[j]}$ ($j = 1, 2, \dots, n$).

Let $\epsilon > 0$ be a small real number. Then,

$$UV^\epsilon = \sum_{j,\ell} e^{i(\theta_j^U + \epsilon\theta_\ell^V)} \langle \phi_j^U | \phi_\ell^V \rangle | \phi_j^U \rangle \langle \phi_\ell^V |. \quad (\text{B1})$$

I now follow the strategy used in Ref. [27] to bound the eigenvalues of UV^ϵ . By writing UV^ϵ in the orthonormal basis $\{|\phi_j^U\rangle\}_{j=1}^n$, I can regard UV^ϵ as a matrix with matrix elements

$$(UV^\epsilon)_{jk} = e^{i\theta_j^U} \sum_{\ell} e^{i\epsilon\theta_\ell^V} \langle \phi_j^U | \phi_\ell^V \rangle \langle \phi_\ell^V | \phi_k^U \rangle. \quad (\text{B2})$$

Let me consider the effect of V^ϵ on the non-degenerate eigenvalues of U first. Suppose $e^{i\theta_a^U}$ is a non-degenerate eigenvalue of the unperturbed unitary matrix U . I define

an invertible matrix F by

$$F_{jk} = \begin{cases} 0 & \text{if } j \neq k, \\ \epsilon\alpha & \text{if } j = k = a, \\ 1 & \text{otherwise,} \end{cases} \quad (\text{B3})$$

where $\alpha > 0$ is a constant to be determined later. Now I can bound the location of the corresponding perturbed eigenvalue of UV^ϵ by applying Gerschgorin circle theorem to the isospectral matrix $FUV^\epsilon F^{-1}$. Specifically, the center C_j and radius R_j of the j th Gerschgorin circle on the complex plane for the matrix $FUV^\epsilon F^{-1}$ are given by

$$C_j = (FUV^\epsilon F^{-1})_{jj} = (UV^\epsilon)_{jj} \quad (\text{B4})$$

and

$$R_j = \sum_{\ell \neq j} |(FUV^\epsilon F^{-1})_{j\ell}| = \begin{cases} \epsilon\alpha \sum_{\ell \neq a} |(UV^\epsilon)_{a\ell}| & \text{if } j = a, \\ \frac{|(UV^\epsilon)_{ja}|}{\epsilon\alpha} + \sum_{\ell \neq a,j} |(UV^\epsilon)_{j\ell}| & \text{if } j \neq a. \end{cases} \quad (\text{B5})$$

Combined with

$$\sum_{\ell=1}^n \langle \phi_j^U | \phi_\ell^V \rangle \langle \phi_\ell^V | \phi_k^U \rangle = \delta_{jk} \quad (\text{B6})$$

where δ_{jk} is the Kronecker delta and

$$|e^{ix} - 1| = 2 \sin \frac{|x|}{2} \leq |x| \text{ whenever } \frac{-\pi}{2} \leq x \leq \frac{\pi}{2}, \quad (\text{B7})$$

I conclude that the distance between the centers of the a th and j th Gerschgorin circles with $j \neq a$ on the complex plane obeys

$$\begin{aligned} |C_a - C_j| &\geq |e^{i\theta_a^U} - e^{i\theta_j^U}| - |(UV^\epsilon)_{aa} - e^{i\theta_a^U}| - |(UV^\epsilon)_{jj} - e^{i\theta_j^U}| \\ &= |e^{i\theta_a^U} - e^{i\theta_j^U}| - \left| e^{i\theta_a^U} \sum_{\ell=1}^n (e^{i\epsilon\theta_\ell^V} - 1) |\langle \phi_a^U | \phi_\ell^V \rangle|^2 \right| - \left| e^{i\theta_j^U} \sum_{\ell=1}^n (e^{i\epsilon\theta_\ell^V} - 1) |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right| \\ &\geq |e^{i\theta_a^U} - e^{i\theta_j^U}| - 2 \sum_{\ell=1}^n |e^{i\epsilon\theta_\ell^V} - 1| \\ &\geq |e^{i\theta_a^U} - e^{i\theta_j^U}| - 2\epsilon \sum_{\ell=1}^n |\theta_\ell^V|. \end{aligned} \quad (\text{B8})$$

And by the same argument, for any $1 \leq x \neq y \leq n$,

$$\begin{aligned} |(UV^\epsilon)_{xy}| &= \left| e^{i\theta_x^U} \sum_{\ell=1}^n e^{i\epsilon\theta_\ell^V} \langle \phi_x^U | \phi_\ell^V \rangle \langle \phi_\ell^V | \phi_y^U \rangle \right| \\ &= \left| \sum_{\ell=1}^n (e^{i\epsilon\theta_\ell^V} - 1) \langle \phi_x^U | \phi_\ell^V \rangle \langle \phi_\ell^V | \phi_y^U \rangle \right| \\ &\leq \sum_{\ell=1}^n |e^{i\epsilon\theta_\ell^V} - 1| \\ &\leq \epsilon \sum_{\ell=1}^n |\theta_\ell^V|. \end{aligned} \quad (\text{B9})$$

Hence, the sum of radii of the a th and j th Gerschgorin

circles satisfies

$$R_a + R_j \leq \left[\frac{1}{\alpha} + \epsilon(n-2) + \epsilon^2 \alpha(n-1) \right] \sum_{\ell=1}^n |\theta_\ell^V|$$

$$< \left[\frac{1}{\alpha} + \epsilon(n-2) + \epsilon^2 \alpha n \right] \sum_{\ell=1}^n |\theta_\ell^V|. \quad (\text{B10})$$

Since $e^{i\theta_a^U}$ is a non-degenerate eigenvalue of the matrix U , I can always find $\alpha, \epsilon > 0$ satisfying

$$\epsilon \alpha \leq \frac{\sqrt{n^2 + 4n} - n}{2n} \quad (\text{B11})$$

and

$$\alpha \geq \max_{j \neq a} \frac{2 \sum_{\ell=1}^n |\theta_\ell^V|}{|\exp(i\theta_a^U) - \exp(i\theta_j^U)|}. \quad (\text{B12})$$

With these choices of α and ϵ , the a th Gerschgorin circle will be disjointed from all the other Gerschgorin circles because $|C_a - C_j| > R_a + R_j$ for all $j \neq a$. So, according to Gerschgorin circle theorem, there is exactly one eigenvalue of UV^ϵ located inside the circle centered at C_a and radius $R_a \leq \epsilon^2 \alpha \sum_{\ell} |\theta_\ell^V| = O(\epsilon^2)$ on the complex plane. Hence, if eigenvalues of U are non-degenerate, the eigenvalues of UV^ϵ for sufficiently small ϵ are given by

$$\sum_{k=1}^n e^{i(\theta_j^U + \epsilon \theta_k^V)} |\langle \phi_j^U | \phi_k^V \rangle|^2 + O(\epsilon^2) \quad (\text{B13})$$

for $j = 1, 2, \dots, n$.

Eq. (B13) also holds even if eigenvalues of U are degenerate. To show this, all I need to do is to modify the above proof as follows. Suppose $e^{i\theta_a^U}$ is an r -fold degenerate eigenvalue of U . And denote \mathcal{H} the r -dimensional Hilbert subspace making up of the corresponding degenerate eigenvectors of U . For ϵ sufficiently close to 0, V^ϵ is strictly diagonally dominant. Hence, the $r \times r$ submatrix $V^\epsilon|_{\mathcal{H}}$ formed by retaining only the rows and columns of V^ϵ corresponding to the span of \mathcal{H} is diagonalizable. In other words, there exists a basis $\{|\phi_j^U\rangle\}_{j=1}^n$ such that $U|_{\mathcal{H}}$ and $V^\epsilon|_{\mathcal{H}}$ are simultaneously diagonalized. Regarding U and V^ϵ as matrices in this basis, I know that

$$\langle \phi_j^U | V^\epsilon | \phi_a^U \rangle = \langle \phi_a^U | V^\epsilon | \phi_j^U \rangle = 0 \text{ if } j \neq a \text{ and } \theta_j^U = \theta_a^U \quad (\text{B14})$$

provided that $\epsilon > 0$ is sufficiently small. I now express UV^ϵ in this basis and replace the diagonal matrix F in Eq. (B3) by

$$F_{jk} = \begin{cases} 0 & \text{if } j \neq k, \\ \epsilon \alpha & \text{if } j = k \text{ and } \theta_j^U = \theta_a^U, \\ 1 & \text{otherwise.} \end{cases} \quad (\text{B15})$$

Then, if $\theta_j^U \neq \theta_a^U$, the inequalities in Eqs. (B8) and (B10) constraining the centers and radii of the a th and j th Gerschgorin circles of the matrix $FUV^\epsilon F^{-1}$ still apply. Whereas in the case of $\theta_j^U = \theta_a^U$,

$$|C_a - C_j|^2 = \left| \sum_{\ell=1}^n \left(|\langle \phi_a^U | \phi_\ell^V \rangle|^2 - |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right) e^{i\epsilon \theta_\ell^V} \right|^2$$

$$= \left[\sum_{\ell=1}^n \left(|\langle \phi_a^U | \phi_\ell^V \rangle|^2 - |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right) \cos(\epsilon \theta_\ell^V) \right]^2 + \left[\sum_{\ell=1}^n \left(|\langle \phi_a^U | \phi_\ell^V \rangle|^2 - |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right) \sin(\epsilon \theta_\ell^V) \right]^2$$

$$= -2 \sum_{\ell \neq \ell'} \left(|\langle \phi_a^U | \phi_\ell^V \rangle|^2 - |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right) \left(|\langle \phi_a^U | \phi_{\ell'}^V \rangle|^2 - |\langle \phi_j^U | \phi_{\ell'}^V \rangle|^2 \right) \sin^2 \left[\frac{\epsilon(\theta_\ell^V - \theta_{\ell'}^V)}{2} \right]. \quad (\text{B16})$$

By Taylor's formula with remainder,

$$|C_a - C_j| = B_j \epsilon + O(\epsilon^3) \quad (\text{B17})$$

where

$$B_j = \left[\frac{-1}{2} \sum_{\ell \neq \ell'} \left(|\langle \phi_a^U | \phi_\ell^V \rangle|^2 - |\langle \phi_j^U | \phi_\ell^V \rangle|^2 \right) \left(|\langle \phi_a^U | \phi_{\ell'}^V \rangle|^2 - |\langle \phi_j^U | \phi_{\ell'}^V \rangle|^2 \right) (\theta_\ell^V - \theta_{\ell'}^V)^2 \right]^{1/2}. \quad (\text{B18})$$

It is important to note that $B_j > 0$ if $C_a \neq C_j$ and that the magnitude of the $O(\epsilon^3)$ remainder term in Eq. (B17) is less than or equal to that of the $B_j \epsilon$ term provided that ϵ is sufficiently close to 0. From Eqs. (B14) and (B15), the radius of the j th Gerschgorin circle of the matrix

$FUV^\epsilon F^{-1}$ obeys

$$R_j \leq \epsilon^2 \alpha(n-r) \sum_{\ell=1}^n |\theta_\ell^V| < \epsilon^2 \alpha n \sum_{\ell=1}^n |\theta_\ell^V| \text{ if } \theta_j^U = \theta_a^U. \quad (\text{B19})$$

Suppose the set $\{j: C_a = C_j\}$ has $r' \leq r$ elements. Then,

by choosing $\alpha, \epsilon > 0$ satisfying Eq. (B11),

$$\alpha \geq \max_{j: \theta_j^U \neq \theta_a^U} \frac{2 \sum_{\ell=1}^n |\theta_\ell^V|}{|\exp(i\theta_a^U) - \exp(i\theta_j^U)|} \quad (\text{B20})$$

and

$$4\epsilon\alpha n \sum_{\ell=1}^n |\theta_\ell^V| \leq \min_{j: \theta_j^U = \theta_a^U \text{ and } B_j \neq 0} B_j, \quad (\text{B21})$$

the r' Gerschgorin circles with a common center C_a and each with $O(\epsilon^2)$ radius will be disjointed from the rest of the Gerschgorin circles. Hence, by Gerschgorin circle theorem, there are exactly r' eigenvalues of UV^ϵ each obeying Eq. (B13). Hence, Eq. (B13) also holds for the degenerate eigenvalue case. I also remark that Eq. (B13) resembles the Rayleigh-Schrödinger series truncated at the ϵ^2 terms for time-independent perturbation of Hermitian operators.

The argument of Eq. (B13) equals $\theta_j^U + \arg \left[\sum_k e^{i\epsilon\theta_k^V} |\langle \phi_j^U | \phi_k^V \rangle|^2 + O(\epsilon^2) \right]$. In order words, the arguments of the eigenvalues of UV^ϵ obey

$$\theta_j^{UV^\epsilon} = \theta_j^U + \epsilon \sum_{k=1}^n \theta_k^V |\langle \phi_j^U | \phi_k^V \rangle|^2 + O(\epsilon^2) \bmod 2\pi. \quad (\text{B22})$$

Since ϵ is a small positive number and all arguments are written in their principle values, Eq. (B22) implies

$$\begin{aligned} |\theta_j^{UV^\epsilon}| &\leq \left| \theta_j^U + \epsilon \sum_{k=1}^n \theta_k^V |\langle \phi_j^U | \phi_k^V \rangle|^2 \right| + O(\epsilon^2) \\ &\leq |\theta_j^U| + \epsilon \sum_{k=1}^n |\theta_k^V| |\langle \phi_j^U | \phi_k^V \rangle|^2 + O(\epsilon^2) \end{aligned} \quad (\text{B23})$$

Note that for sufficiently small $\epsilon > 0$, the equality in the first line holds if and only if $|\theta_j^U + \epsilon \sum_k \theta_k^V |\langle \phi_j^U | \phi_k^V \rangle|^2| \leq \pi$. Now, applying the eigenvalue perturbation and stability results for the sum of two diagonalizable matrices in Ref. [27], I conclude that the eigenvalues of the positive semi-definite Hermitian matrix

$$\begin{aligned} &\tilde{H}(U, V, \epsilon) \\ &\equiv \sum_{j=1}^n (|\theta_j^U| |\phi_j^U\rangle \langle \phi_j^U| + \epsilon |\theta_j^V| |\phi_j^V\rangle \langle \phi_j^V|) \\ &= \sum_{j=1}^n \left[|\theta_j^U| |\phi_j^U\rangle \langle \phi_j^U| + \epsilon |\theta_j^V| |\phi_j^V\rangle \langle \phi_j^V| \right] \\ &\equiv \tilde{H}_a(U) + \epsilon \tilde{H}_b(V) \end{aligned} \quad (\text{B24})$$

are precisely those given in the last line of Eq. (B23). (Similar to the above analysis for the degenerate eigenvalue case, this result is also true when eigenvalues of $\tilde{H}_a(U)$ are degenerate. The key of the proof is to carefully pick a basis so that $\tilde{H}_a(U)$ and $\tilde{H}_b(V)$ are simultaneously diagonalized for every degenerate subspace of $\tilde{H}_a(U)$.) Combined with Eq. (B23), I conclude that

$$|\theta_j^\downarrow(UV^\epsilon)| \leq \lambda_j^\downarrow(\tilde{H}(U, V, \epsilon)) + O(\epsilon^2) \quad (\text{B25})$$

for $j = 1, 2, \dots, n$.

According to Corollary 6.6 in Ref. [14],

$$\begin{aligned} \sum_{j=1}^m |\theta_j^\downarrow(UV^\epsilon)| &\leq \sum_{j=1}^m \lambda_j^\downarrow(\tilde{H}(U, V, \epsilon)) + O(\epsilon^2) \\ &\leq \sum_{j=1}^m \left[\lambda_j^\downarrow(\tilde{H}_a(U)) + \lambda_j^\downarrow(\epsilon \tilde{H}_b(V)) \right] + O(\epsilon^2) \\ &= \sum_{j=1}^m \left[|\theta_j^\downarrow(U)| + \epsilon |\theta_j^\downarrow(V)| \right] + O(\epsilon^2). \end{aligned} \quad (\text{B26})$$

Interestingly, this inequality is the Hermitian analogy of what I need to prove here.

Now, by iteratively applying Eq. (B26) to $UV = \overbrace{U V^{1/q} V^{1/q} \dots V^{1/q}}^{q \text{ terms}}$, I get

$$\sum_{j=1}^m |\theta_j^\downarrow(UV)| \leq \sum_{j=1}^m \left[|\theta_j^\downarrow(U)| + |\theta_j^\downarrow(V)| \right] + O\left(\frac{1}{q}\right). \quad (\text{B27})$$

By taking the limit $q \rightarrow +\infty$, I obtain the second inequality in Eq. (16). \square

Remark 4. Note that for each $j = 1, 2, \dots, n$, the equality of Eq. (B23) holds if and only if $\theta_j^{UV^\epsilon}$, θ_j^U and θ_k^V are all non-negative or non-positive for all k whenever $\langle \phi_j^U | \phi_k^V \rangle \neq 0$. And from the proof of Corollary 6.6 in Ref. [14], the equality of Eq. (B26) holds if and only if the spans of $\{|\xi_j^\downarrow(\tilde{H}(U, V, \epsilon))\rangle\}_{j=1}^m$, $\{|\xi_j^\downarrow(\tilde{H}_a(U))\rangle\}_{j=1}^m$ and $\{|\xi_j^\downarrow(\tilde{H}_b(V))\rangle\}_{j=1}^m$ agree. Suppose the vector $\vec{\mu}$ consists of r distinct μ_j 's. Then using the above two observations and by induction on r , one can prove the following necessary and sufficient conditions for the second inequality in Eq. (16) of Theorem 2 to become an equality. The detailed proof is left to interested readers.

1. The n -dimensional Hilbert space \mathcal{H} on which U, V and UV act can be written as the direct sum $\bigoplus_{j=1}^r \mathcal{H}_j$. Moreover U, V and UV are simultaneously block diagonalized with respected to this direct sum decomposition of \mathcal{H} . That is to say, $\langle \phi | U | \psi \rangle = 0$ whenever $|\phi\rangle$ and $|\psi\rangle$ belong to different \mathcal{H}_j 's. And similarly for V and UV .
2. The ordering of absolute values of the arguments of eigenvalue of U, V and UV respects the direct sum decomposition of \mathcal{H} in the sense that $|\theta_k^\downarrow(U)|_{\mathcal{H}_j} \geq |\theta_{k'}^\downarrow(U)|_{\mathcal{H}_{j'}}$ for all k, k' whenever $j > j'$. And similarly for V and UV . Furthermore, when calculating $\nu(\cdot)_{\vec{\mu}}$ using Eq. (7), the absolute values of the arguments of the eigenvalue in each of the corresponding diagonal blocks in U, V and UV are associated with the same value of μ_j .
3. If the μ_j corresponding to \mathcal{H}_j is non-zero, then $U|_{\mathcal{H}_j}$, $V|_{\mathcal{H}_j}$ and $UV|_{\mathcal{H}_j}$ can be further simultaneously block diagonalized with respected to the direct

sum decomposition $\mathcal{H}_j = \mathcal{H}_j^+ \oplus \mathcal{H}_j^-$. Furthermore, $\theta_k^\downarrow(U|_{\mathcal{H}_j^+})$, $\theta_k^\downarrow(V|_{\mathcal{H}_j^+})$ and $\theta_k^\downarrow(UV|_{\mathcal{H}_j^+}) \geq 0$ for all k ; while $\theta_k^\downarrow(U|_{\mathcal{H}_j^-})$, $\theta_k^\downarrow(V|_{\mathcal{H}_j^-})$ and $\theta_k^\downarrow(UV|_{\mathcal{H}_j^-}) \leq 0$ for all k .

Since the number of independent conditions for equality in Eq. (16) increases with the dimension n of the unitary matrices as well as the value m used in $\nu(\cdot)_{\vec{\mu}^{[m]}}$, I conclude that as n or m increases, it is harder and harder for the equality to hold provided that the unitary matrices are drawn randomly from the Haar measure of $U(n)$. I

verify this assertion in Fig. 1, which shows the plots of $\nu(UV)_{\vec{\mu}^{[m]}}$ against $\nu(U)_{\vec{\mu}^{[m]}} + \nu(V)_{\vec{\mu}^{[m]}}$.

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- [1] I. Bengtsson and K. Życzkowski, *Geometry Of Quantum States* (CUP, Cambridge, U.K., 2006).
 - [2] A. Ashikhmin, S. Litsyn, and M. A. Tsfasman, *Phys. Rev. A*, **63**, 032311 (2001).
 - [3] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, *Science*, **311**, 1133 (2006).
 - [4] N. Johnston and D. W. Kribs, *J. Math. Phys.*, **51**, 082202 (2010).
 - [5] N. Johnston and D. W. Kribs, *Quant. Inform. & Comp.*, **11**, 104 (2011).
 - [6] A. E. Rastegin, *Quant. Inf. Proc.*, **9**, 61 (2010).
 - [7] A. E. Rastegin, *Quant. Inf. Proc.*, **10**, 123 (2011).
 - [8] M. A. Nielsen and I. L. Chuang, *Quantum Computation And Quantum Information* (CUP, Cambridge, U.K., 2000) Chap. 9.
 - [9] A. A. Nudel'man and P. A. Švarcman, *Uspehi Math. Nauk.*, **13**, 111 (1958).
 - [10] R. C. Thompson, *Lin. & Multilin. Alg.*, **2**, 13 (1974).
 - [11] S. Agnihotri and C. Woodward, *Math. Res. Lett.*, **5**, 817 (1998).
 - [12] A. M. Childs, J. Preskill, and J. Renes, *J. Mod. Opt.*, **47**, 155 (2000).
 - [13] R. Bhatia, *Matrix Analysis* (Springer, Berlin, Germany, 1997).
 - [14] R. Bhatia, *Perturbation Bounds For Matrix Eigenvalues* (SIAM, Philadelphia, USA, 2007).
 - [15] A. Knutson and T. Tao, *Notices A. M. S.*, **48**, 175 (2001).
 - [16] S. Lloyd, *Nature*, **406**, 1047 (2000).
 - [17] H. F. Chau, *Phys. Rev. A*, **81**, 062133 (2010).
 - [18] N. Margolus and L. B. Levitin, in *Proceedings of the fourth workshop on physics and computation*, edited by T. Toffoli, B. Biafore, and J. Leão (New England Complex Systems Institute, Boston, U.S.A., 1996) pp. 208–211.
 - [19] N. Margolus and L. B. Levitin, *Physica D*, **120**, 188 (1998).
 - [20] A. Uhlmann, *Phys. Lett. A*, **161**, 329 (1992).
 - [21] F. Curcker and A. G. Corbalan, *Amer. Math. Monthly*, **96**, 342 (1989).
 - [22] E. E. Tyrtysnikov, *A Brief Introduction To Numerical Analysis* (Birkhäuser, Boston, U.S.A., 1997) Theorem 3.9.1.
 - [23] C.-K. Li, T.-Y. Tam, and N.-K. Tsing, *Lin. & Multilin. Alg.*, **16**, 215 (1984).
 - [24] Y. T. Lam, *Quantum information*, Final year project report (Dept. of Physics, Univ. of Hong Kong, 2009).
 - [25] H. F. Chau and Y. T. Lam, *J. Inequal. & Appl.* (2011), to appear, arXiv:1006.3978.
 - [26] H. F. Chau, “Metrics on unitary matrices, bounds on eigenvalues of product of unitary matrices, and measures of non-commutativity between two unitary matrices,” (2010), arXiv:1006.3614.
 - [27] K. E. Atkinson, *An Introduction To Numerical Analysis* (Wiley, N.Y., U.S.A., 1978) pp. 509–512, in particular Eq. (9.27).